ON GREEN FUNCTIONS IN $\mathbf{R}^2$

BY

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ABSTRACT

Conditions which guarantee the validity of the use of Lord Kelvin's method of electrostatic images to find Green functions of the unbounded Green domains in the plane are examined.

Introduction

One way of defining the Green function of a bounded domain D is

$$G_D(x, y) = \int \log|z - y| \, P_D(x, dz) - \log|x - y|$$

where $P_D(x, dz)$ is the harmonic measure at x with respect to the domain D. For unbounded open sets the situation is more complicated. An unbounded open set may not possess a Green function and even when it does possess one, this function cannot in general be defined by the above "formula". The complement of the unit disc is a case in point; it is a Green domain but its Green function does not obey the above formula.

In some physics and mathematical physics books the Green functions of certain — admittedly simple — domains are found by the ingenious method of "electrostatic images" due to Lord Kelvin. The method of images consists in guessing, using some symmetry of the domain, a simple distribution of charges outside the domain, whose potential on the boundary equals that due to a point charge inside the domain. One then uses the above formula to compute the Green function. This is the case for example with the upper half plane, a wedge, a strip, a quadrant etc. It seems to us that two assumptions are implicit in this line of reasoning: first, that the above formula is valid and, second, that the outer distribution of charges and the charge induced on the boundary due to the point charge inside give rise to the same field inside the domain. Physically perhaps this is obvious, but no mathematical reason is given. Mathematically

Received February 6, 1974

speaking, these two assumptions warrant justification, since we are dealing with unbounded domains and harmonic functions with unbounded boundary values. It is probably not out of place to add that in each of the above cases one can justify this procedure.

One of the main results of this paper is Theorem 10 which gives a necessary and sufficient condition for the validity of the above formula. It will transpire that the Physicists are entirely justified in their assumptions.

Our methods are probabilistic and the language of Brownian motion will be used throughout.

Notation

x_t will denote the standard Brownian motion in $\mathbf{R}^2$. For $x \in \mathbf{R}^2$, $|x|$ will denote its norm. For an open set D the *first exit time after zero from D*, notation T_D or, for simplicity, T is defined by

$$T = \inf(t : t > 0\ x_t \notin D).$$
$$= \infty \text{ if no such } t \text{ exists.}$$

P_x, E_x denote the probability and expectation corresponding to the starting point x. For T as above it is well known that

$$P_x(T = 0) = 1 \text{ or } 0$$

for every $x \in \mathbf{R}^2$. A point $x \in \partial D$ will be called *regular* if

$$P_x(T = 0) = 1.$$

Otherwise it is called *irregular*. The set of regular points will be denoted by $r(D)$.

Preliminaries

PROPOSITION 1. *If s is superharmonic in an open set G, D a bounded open set with $\bar{D} \subset G$, T = exit time after zero from D, then*

$$u(x) = E_x[s(x_T)],\ x \in G$$

is superharmonic in G, harmonic in D, $u \leqq s$ and

$$u(x) = s(x) \text{ if } x \in r(D) \cup (G \setminus \bar{D}).$$

In brief the outline of a proof is: Superharmonicity is a local property. Since s is bounded below on a neighbourhood of $\bar{D}$, one need only consider the case

$s \geqq 0$ on G. Because s is then an increasing limit of bounded continuous superharmonic (on G) functions one assumes that s is bounded and continuous. $s(x_t)1_{t<S}$ is then a continuous non-negative supermartingale where $S =$ exit time from G. That $T = \lim(t + T(\theta_t))$ with θ_t the shift operator $x_{t_1}(\theta_t) = x_{t_1+t}$ and the optional sampling theorem for supermartingales can be used to show that u is excessive, i.e., that u is superharmonic on G.

Let us remark in passing that boundedness of D is not needed in Proposition 1 if s is assumed non-negative; this is clear from the above outline. Proposition 1 is valid in any dimension.

PROPOSITION 2. *The set of irregular points in ∂D has 2-dimensional Lebesgue measure zero.*

The proof is given below. In fact the set of irregular points has capacity (logarithmic) zero; this fact, known as Kellog-Evans theorem, is however much harder to prove than Proposition 2.

We shall write

$$K(x) = \log \frac{1}{|x|}, \qquad x \neq 0 \tag{1}$$

$$= \infty \qquad , \qquad x = 0.$$

K is superharmonic on $\mathbf{R}^2$ and is harmonic in the complement of the origin.

PROPOSITION 3. *Let D be a bounded open set and T = first exit time after zero from D. For all $x, y \in \mathbf{R}^2$*

$$E_x[K(x_T - y)] = E_y[K(x_T - x)]. \tag{2}$$

PROOF. Fix $x_0 \in D$. Let u and v denote the left and right side of (2) with x replaced by x_0. u and v are superharmonic on $\mathbf{R}^2$ and harmonic in D. That u is superharmonic is a consequence of Fubini's theorem and the superharmonicity of K while the superharmonicity of v is a consequence of Proposition 1. Let B_n be a sequence of open sets with $x_0 \in B_1$, $\bar{B}_n \subset B_{n+1}$ and union $\cup B_n = D$. Denote by T_n the exit time after zero from B_n. u is harmonic in D and D is a neighbourhood of $\bar{B}_n$. For $y \in \bar{B}_n$ then

$$u(y) = E_y[u(x_{T_n})], \qquad y \in \bar{B}_n. \tag{3}$$

For all z, $u(z) \leqq K(x_0 - z)$, because for fixed z, $K(\cdot - z)$ is superharmonic and Proposition 1 applies. From (3):

(4) $$u(y) = E_y[u(x_{T_n})] \leqq E_y[K(x_{T_n} - x_0)], \qquad y \in \bar{B}_n.$$

Letting n tend to infinity in (4), by bounded convergence (recall $x_0 \in B$, and $x_{T_n} \in \partial B_n$):

(5) $$u(y) \leqq E_y[K(x_T - x_0)] = v(y), \qquad y \in D.$$

$x_0 \in D$ is arbitrary, by symmetry, (5) implies (2) for all $x, y \in D$. In particular,

(6) $$u(y) = v(y), \qquad y \in D.$$

Write $s_n(y) = E_{x_0}[K(x_{T_n} - y)]$. s_n are superharmonic and decrease as n increases. If f is bounded Borel with compact support, $\int K(z-y)f(y)dy$ is a continuous function of z on $\mathbf{R}^2$. (see [2] pp. 119).

$$\int u(y)f(y)dy = E_{x_0}\left[\int K(x_T - y)f(y)dy\right]$$

$$= E_{x_0}\left[\lim_n \int K(x_{T_n} - y)f(y)dy\right]$$

since x_0 being in D, $P_{x_0}(T_n \uparrow T) = 1$,

$$= \lim_n \int f(y)s_n(y)dy = \int f(y)\lim_n s_n(y)dy.$$

Showing that $u(y) = \lim s_n(y)$ for almost all y. If $y \notin D$, $s_n(y) = K(x_0 - y)$ because $K(\cdot - y)$ is then harmonic in D which is a neighbourhood of $\bar{B}_n$. In particular, we see that $u(y) = K(x_0 - y)$ for almost all $y \notin D$.

Now let us digress and prove Proposition 2. If $b \in \partial D$ is irregular, the Brownian path starting at b spends a positive time in D and this time must be spent in a connected component of D. This means that if $b \in \partial D$ is irregular, it must be the boundary point of a connected component of D. We may thus assume D is connected. We claim then $K(x_0 - b) - u(b) > 0$. Otherwise by lower semi-continuity of u,

$$\limsup_{D \ni y \to b}[K(x_0 - y) - u(y)] \leqq K(x_0 - b) - u(b) = 0.$$

$K(x_0 - \cdot) - u(\cdot)$ is therefore a "barrier" at b and b is regular (see Proposition 11). Since $u(y) = K(x_0 - y)$ for almost all $y \notin D$, we have shown Proposition 2.

We continue with the proof of Proposition 3. Clearly $K(x_0 - y) = v(y)$ for $y \notin \bar{D}$ or if $y \in \partial D$ is regular. The set of irregular points in ∂D having measure zero, $u(y)$ thus equals $v(y)$ for almost all $y \notin D$ and together with (6), $u(y) = v(y)$ for almost all $y \in \mathbf{R}^2$. This is valid for any $x_0 \in D$. We have thus shown

(7) $$E_x[K(x_T - y)] = E_y[K(x_T - x)], \quad x \in D, \quad y \in \mathbf{R}^2.$$

Now suppose $x = x_0 \in \partial D$ and regular. The left side of (2) is then $\equiv K(x_0 - y)$. If $y \in D$, (7) implies $E_y[K(x_T - x_0)] = E_{x_0}[K(x_T - y)] = K(x_0 - y)$. If $y \notin \bar{D}$ or in ∂D and regular, the right side of (2) equals $K(x_0 - y)$. Thus if $x = x_0 \in \partial D$ and regular $E_{x_0}[K(x_T - y)] = E_y[K(x_T - x_0)]$ except perhaps when $y \in \partial D$ and irregular. Proposition 2 forces identity:

(8) $$E_x[K(x_T - y)] = E_y[K(x_T - x)], \quad x \in D \cup r(D), \quad y \in \mathbf{R}^2.$$

For a fixed $y \in \mathbf{R}^2$, by (8) the left and right sides of (2) coincide except perhaps for $x \in \partial D$ that are irregular. One more appeal to Proposition 2 and we are done. That proves the proposition.

Remark 1. In $\mathbf{R}^d$, $d \geqq 3$, the fundamental superharmonic functions are $|x|^{-d+2}$. The same proof shows that Proposition 3 is valid for $\mathbf{R}^d$ for any d. If $d \geqq 3$ one need not assume that D is bounded.

Remark 2. For an open set D having a Green function, Hunt in [3] shows that a reasonable version $q(t,x,y)$ of a density of the measure $P_x[x_t \in dy,\ t < T]$ can be chosen, such that $q(t,x,y)$ is symmetric in x and y for all x, y. He then puts $G(x,y) = \int_0^\infty q(t,x,y)dt$, shows that it is finite if $x \neq y$ and that its restriction to $D \times D$ is the Green function of D. It seems to us that to deduce Proposition 3 from this result we must go through the proof of Proposition 3; this is due to the occurance of infinities. On the other hand, consider the following: Fix $\alpha > 0$. A non-negative function s is called α-excessive iff $e^{-\alpha t} E_x[s(x_t)] \leqq s(x)$ for all t and $\lim_{t \to 0} E_x[s(x_t)] = s(x)$. We see at once from this definition that two α-excessive functions equal almost everywhere are identical. If D is open, T = exit time after zero from D and s α-excessive, so is $u(x) = E_x[e^{-\alpha T} s(x_T)]$. Now let for $x \in \mathbf{R}^d$,

$$L(x) = \int_0^\infty e^{-\alpha t}(2\pi t)^{-d/2} e^{-|x|^2/2t} dt = 2(2\pi)^{-d/2}\left(\frac{\sqrt{2\alpha}}{|x|}\right)^{d/2-1} K_{d/2-1}(\sqrt{2\alpha}|x|),$$

where K denotes the modified Bessel function of the third kind. L is α-excessive and off the origin $\Delta L = 2\alpha L$ where Δ is the Laplacian. Simple adaptation of the proof of Proposition 3 shows that for all x, y,

$$E_x[e^{-\alpha T} L(x_T - y)] = E_y[e^{-\alpha T} L(x_T - x)].$$

Using this, it is not difficult to show that $q(t,x,y)$ is symmetric in x and y where

$q(t, x, y)$ is an upper semi continuous version of the density of $P_x[x_T \in dy,\ t < T]$. We omit the details.

We shall need a corollary of Proposition 3:

COROLLARY 4. *Let D be bounded open and $x_0 \in D$. If $y_0 \in \partial D$ is regular*

$$\lim_{D \ni y \to y_0} E_{x_0}[K(x_T - y)] = K(x_0 - y_0). \tag{9}$$

If $y_0 \in \partial D$ is irregular

$$\liminf_{D \ni y \to y_0} E_{x_0}[K(x_T - y)] = E_{x_0}[K(x_T - y_0)]. \tag{10}$$

PROOF. (9) follows from (2) if we note that $K(\cdot - x_0)$ is continuous on ∂D. (10) follows from the following more general proposition.

PROPOSITION 5. *Let s be superharmonic in an open set G and D a bounded open set with $\bar{D} \subset G$. For each $y_0 \in \partial D$ that is irregular*

$$\liminf_{D \ni y \to y_0} s(y) = s(y_0).$$

PROOF. We need only restrict ourselves to a neighbourhood of $\bar{D}$ and therefore we may assume that s is excessive in G. If T denotes the exit time after zero from D,

$$s(y_0) \geqq E_{y_0}[s(x_{T \wedge t}) : t < S]$$

where $S =$ exit time from G. Letting $t_n \to 0$,

$$s(y_0) \geqq E_{y_0}\left| \liminf_{t_n \to 0} s(x_{t_n}) 1_{t_n < T} \right| \geqq s(y_0).$$

For $t_n < T$, $s(x_{t_n}) \in D$. That completes the proof.

We omit the proof of the following simple proposition. K^+ and K^- denote: $K^+ = K \vee 0$, $-K^- = K \wedge 0$.

PROPOSITION 6. *Let m be a finte measure on the plane. Then either $\int |K(\cdot - y)| m(dy) \equiv \infty$ or $\int |K(\cdot - y)| m(dy)$ is locally integrable. In the latter case $\int K^-(x - y) m(dy)$ is finite for all x, $\int K(\cdot - y) m(dy)$ is superharmonic on the plane, harmonic off the support of m, and for any x, $\int K(x - y) m(dy) = \infty$ iff $\int |K(x - y)| m(dy) = \infty$.*

Let D be an unbounded open set and D_n an increasing sequence of bounded open sets with union D. The Green functions of D_n increase to a limit which is either $\equiv \infty$ on D or is finite off the diagonal of $D \times D$. In the latter case we say that D is Greenian or that D has a Green function and the limit which is independent of the exhausting sequence D_n is called the Green function of D. It can be shown that D is Greenian iff $P_x(T < \infty) = 1$ for all $x \in \mathbf{R}^2$ where T is the first exit time after zero from D. All the open sets considered in this paper will be assumed to have Green functions.

Given D, let $D_n = B_n \cap D$ where B_n is the disc of radius n, centre 0: $B_n = (x : |x| < n)$. Put

$$s_n(x, y) = E_x[K(x_{T_n} - y)],\ x, y \in \mathbf{R}^2,\ T_n = \text{exit time from } D_n. \tag{11}$$

$s_n(x, \cdot)$ are superharmonic on $\mathbf{R}^2$ and decrease. Let

$$s(x, y) = \lim s_n(x, y) \tag{12}$$

$s(x, y)$ may *a priori* be $-\infty$. If $x \notin \bar{D}$ or $x \in \partial D$ is regular $s(x, y) = s_n(x, y) = K(x - y)$ for all y. The only interesting cases are $x \in D$ and $x \in \partial D$ is irregular. It will transpire that $s(x, y) > -\infty$ in these cases also. s is symmetric because the s_n are (Proposition 3).

Suppose $x_0 \in D$, $y_0 \in D$. D being Greenian $s(x_0, y_0) > -\infty$. From an n on $|x_{T_n} - y_0|$ is clearly bounded below and $K(x_{T_n} - y_0)$ is then bounded above. Fatou Lemma is applicable:

$$\begin{aligned} -\infty < s(x_0, y_0) &\leqq E_{x_0}[\limsup K(x_{T_n} - y_0)] \\ &= E_{x_0}[K(x_T - y_0)]. \end{aligned}$$

$K(x_T - y_0)$ being bounded above, $E_{x_0}[K(x_T - y_0)]$ makes sense and from the last inequality we deduce that $E_{x_0}[|K(x_T - y_0)|] < \infty$. By Proposition 6, $E_{x_0}[K^-(x_T - y)]$ is finite for all y. For any y then

$$\begin{aligned} E_{x_0}[K(x_{T_n} - y)] = E_{x_0}[K^+(x_T - y) : T_n = T] \\ - E_{x_0}[K^-(x_T - y) : T_n = T] \\ + E_{x_0}[K(x_{T_n} - y) : T_n < T]. \end{aligned} \tag{13}$$

$P_{x_0}(T_n < T)$ tends to zero. On the set $T_n < T$, $|x_{T_n}| = n$ and

$$\frac{|x_{T_n} - y|}{n}$$

tends boundedly to 1. Putting $y = y_0$ in (13) and taking limits shows (since

$E_{x_0}[|K(x_T - y_0)|] < \infty$ and $s(x_0, y_0) > -\infty$ that $\lim_{n\to\infty}(\log n)\cdot P_{x_0}(T_n < T)$ exists and is finite. For any y then

$$s(x_0, y) = E_{x_0}[K^+(x_T - y)] - E_{x_0}[K^-(x_T - y)]$$
$$- \lim(\log n)P_{x_0}(T_n < T).$$

The left side of the last equality can at worst be $-\infty$ whereas the right side at worst $+\infty$. One concludes that $s(x_0, y) > -\infty$ for all y and $E_{x_0}[|K(x_T - y)|] < \infty$ for all y. s is symmetric. So $s(x, y) > -\infty$ for all x provided $y \in D$.

Let $x_0 \in \partial D$ be irregular and $y_0 \in D$. $s(x_0, y_0) > -\infty$. As before (since $|x_{T_n} - y_0|$ is bounded below), $E_{x_0}[|K(x_T - y_0)|] < \infty$. Repetition of the arguments in the last paragraph show that $E_{x_0}[|K(x_T - y)|] < \infty$ for all y and $s(x_0, y) > -\infty$ for all y. We state all this in

LEMMA 7. *With the above notation*

$$s(x, y) = E_x[K(x_T - y)] - a(x),\ x, y \in \mathbf{R}^2$$

(14)

$$a(x) = \lim_{n\to\infty} (\log n)P_x(T_n < T),\ x \in \mathbf{R}^2$$

If $x \in D$ or $x \in \partial D$ is irregular, $E_x[|K(x_T - y)|] < \infty$ for all y and

$$\lim E_x[|K(x_{T_n} - y)|] = E_x[|K(x_T - y)|] + a(x), \quad x, y \in \mathbf{R}^2$$

where $a(x)$ is defined in (14).

To continue, let $x_0 \in D$ or $x_0 \in \partial D$ and irregular. We have seen that $E_{x_0}[K(x_T - \cdot)]$ is a finite lower semi continuous function on $\mathbf{R}^2$. The expression

$$G(x_0, y) = K(x_0 - y) - s(x_0, y) \tag{15}$$
$$= K(x_0 - y) - E_{x_0}[K(x_T - y)] + a(x_0)$$

makes sense, is finite everywhere except at $y = x_0$, is non-negative (since $K \geqq s_n$ and $s \leqq s_n$) and is upper semi continuous. We now show that outside of every neighbourhood of x_0 it is a bounded function of y. Because of upper semi-continuity, it is enough to show that it is bounded at ∞. $|x_T - y| \leqq |x_T - x_0| + |x_0 - y|$. Since $E_{x_0}[|K(x_T - x_0)|] < \infty$, $\log(|x_T - y|)/(|x_0 - y|)$ is bounded above by an integrable function and tends to 1 as $y \to \infty$. By Fatou's lemma,

$$\limsup_{y\to\infty} \{K(x_0 - y) - E_{x_0}[K(x_T - y)]\} \leqq 0, \tag{16}$$

proving that $\limsup_{y\to\infty} G(x_0, y) \leqq a(x_0)$.

LEMMA 8. *Let $x_0 \in D$ or $x_0 \in \partial D$ be irregular. $s_n(x_0, y)$ converges uniformly on compact sets to*

$s(x_0, y) = E_{x_0}[K(x_T - y)] - a(x_0)$.

PROOF. For all m and z,

$$0 \leqq K(x_0 - z) - s_m(x_0, z) \leqq K(x_0 - z) - s(x_0, z)$$
$$= G(x_0, z).$$

By strong Markov property and Proposition 3

$$\begin{aligned} s_n(x_0, y) - s_m(x_0, y) & \\ &= E_y[K(x_T - x_0) - E_{x_{T_n}}[K(x_{T_m} - x_0] : T_n < T_m] \\ &= E_y[K(x_{T_n} - x_0) - s_m(x_0, x_{T_n}) : T_n < T_m] \\ &\leqq E_y[G(x_0, x_{T_n}) : T_n < T]. \end{aligned}$$

On the set $T_n < T$, $|x_{T_n}| = n$. We have seen that away from x_0, G is bounded. It is easy to show that $P_y(T_n < T)$ tends to zero uniformly on compact sets. That proves the lemma.

Lemma 8 and Corollary 4 give

COROLLARY 9. *For $x \in \mathbf{R}^2$ and $b \in \partial D$*

$$\liminf_{D \ni y \to b} E_x[K(x_T - y)] = E_x[K(x_T - b)]. \tag{17}$$

If $b \in \partial D$ is regular the lim inf in (17) can be replaced by a limit. If $b \not\in \bar{D}$ or $b \in \partial D$ and is regular

$$E_x[K(x_T - b)] = K(x - b) + a(x). \tag{18}$$

Clearly Corollary 9 says nothing if $x \not\in \bar{D}$ or $x \in \partial D$ and is regular.

It is possible to show that: a is non-negative harmonic in D, $\lim_{D \ni x \to b} a(x) = 0$ if $b \in \partial D$ is regular,

$$a(x) \leqq 2 \log |x| + O(1)$$

$$\begin{aligned} a(x) &= -\liminf_{y\to\infty}\{K(x-y) - E_x[K(x_T - y)]\} \\ &= \limsup_{y\to\infty} G(x, y). \end{aligned}$$

Applications

In very simple cases (18) can be used to find a. If D is the complement of the closed unit disc and $b=0$ we get at once from (18) $a(x)=\log|x|$ for $|x|>1$. If D is the upper half plane, x_T lies on the real axis. For any b of the form $b=(0,-n)$ and z real, clearly $|z-b|\geqq n$. It follows that $E_x[K(x_T-b)]\leqq \log 1/n$. Using (18) and letting $n\to\infty$, we get $a(x)\leqq 0$, i.e., $a=0$. Consider again the strip $(0<\operatorname{Im} z<1)=D$. For $z\in\partial D$, $|z-n|\geqq n-1$ so that for $x\in D$, $E_x[K(x_T-n)]\leqq \log(1/(n-1))$. Use (18) and let $n\to\infty$ to get $a\equiv 0$. Similar arguments can be used to show that $a=0$ for a wedge, a quadrant, etc. One may expect that simple connectivity of a Green domain is sufficient to guarantee that $a\equiv 0$. This will turn out to be correct.

Again in some simple cases, (18) can be used to compute the harmonic measure at x, i.e., the distribution of x_T relative to P_x. As an example, consider D = the upper half plane. Denote the points on the plane by (x,y). If $b>0$ the point $(a,-b)\notin \bar{D}$. From (18):

$$\int \log[(z-a)^2+b^2]P_{(x,y)}(dz)=\log[(x-a)^2+(y+b)^2],$$

where $P_{(x,y)}(dz)=P_{(x,y)}(x_T\in dz)$. Differentiate both sides relative to b:

$$\int \frac{b}{(a-z)^2+b^2}\,P_{(x,y)}(dz)=\frac{y+b}{(x-a)^2+(y+b)^2}\,.$$

Multiply both sides by $e^{i\alpha a}$ and integrate relative to da from $-\infty$ to ∞:

$$e^{-|\alpha|b}\,\hat{P}_{(x,y)}(\alpha)=e^{i\alpha x}\,\bar{e}^{|\alpha|(y+b)}$$

where $\hat{P}_{(x,y)}(\alpha)=\int e^{i\alpha z}P_{(x,y)}(dz)$. Inversion shows that $P_{(x,y)}(dz)$ has density

$$\frac{1}{\pi}\,\frac{y}{(x-z)^2+y^2}\,.$$

We have thus found the Poisson kernel for the half plane $((x,y):y>0)$.

As another example, consider the strip $((x,y):0<x<1)=D$. Fix $(x,y)\in D$. $P_{(x,y)}(x_T\in dz)$ lives on the lines $x=0$ and $x=1$. Denote these parts by Q_0 and Q_1 respectively. If $a>1$ the point $(a,b)\notin\bar{D}$. From (18):

$$\int \log[a^2+(b-z)^2]Q_0(dz)+\int \log[(a-1)^2+(b-z)^2]Q_1(dz)$$
$$=\log[(x-a)^2+(y-b)^2].$$

As before differentiating relative to a then multiplying both sides by $e^{i\alpha b}$ and integrating relative to db from $-\infty$ to ∞:

$$\hat{Q}_0(\alpha) + e^{|\alpha|} \hat{Q}_1(\alpha) = e^{|\alpha| x} e^{i\alpha y}$$

where $\hat{Q}_j(\alpha) = \int e^{i\alpha z} Q_j(dz)$, $j = 0, 1$.

Also the point $(-a, b) \notin \bar{D}$ if $a > 0$. Repetition of the above leads to:

$$Q_0(\alpha) + e^{-|\alpha|} \hat{Q}_1(\alpha) = e^{-|\alpha| x} e^{i\alpha y}.$$

We must thus have

$$e^{|\alpha|}[1 - e^{-2|\alpha|}] \hat{Q}_1(\alpha) = e^{i\alpha y}[e^{|\alpha| x} - e^{-|\alpha| x}]$$

which expands into

$$\hat{Q}_1(\alpha) = \sum_0^\infty e^{i\alpha y}\{e^{-(2n+1-x)|\alpha|} - e^{-(2n+1+x)|\alpha|}\}.$$

$Q_1(dz)$ thus has density

$$\frac{1}{\pi} \sum_0^\infty \left\{ \frac{2n+1-x}{(2n+1-x)^2 + (y-z)^2} - \frac{2n+1+x}{(2n+1+x)^2 + (y-z)^2} \right\}$$

and $Q_0(dz)$ has density

$$\frac{1}{\pi} \sum_0^\infty \left\{ \frac{2n+x}{(2n+x)^2 + (y-z)^2} - \frac{2n+2-x}{(2n+2-x)^2 + (y-z)^2} \right\}.$$

The above can be interpreted in the "image method".

The main result

Let D be an unbounded connected open set having a Green function. We shall call ∞ *regular relative to* D if $\lim_{D \ni y \to \infty} G_D(x_0, y) = 0$ for some (and hence for all) $x_0 \in D$ where G_D the Green function for D is given by

$$G_D(x, y) = K(z - y) - E_x[K(x_T - y)] + a(x), \quad x, y \in D$$

with a defined in (14).

THEOREM 10. *Let D be an unbounded Green domain.*

$$G_D(x, y) = K(x - y) - E_x[K(x_T - y)] \tag{19}$$

iff ∞ is regular relative to D. Further if ∞ is regular relative to D, for each $y \in D$ or $y \in \partial D$ regular

$$K(x - y) = E_x[K(x_T - y)], \quad x \in \mathbf{R}^2. \tag{19)'$$

PROOF. Suppose (19) holds. We must show that for each $x_0 \in D$, $\lim G_D(x_0, y) = 0$ as y tends to ∞ in D. From (16) $\limsup_{D \ni y \to \infty} G_D(x_0, y) \leqq 0$. On the other hand, if (19) is valid, $\liminf_{D \ni \to \infty} G_D(x_0, y) \geqq 0$, (since $G_D(x_0, y) \geqq 0$). That proves that ∞ must be regular.

In the other direction assume that ∞ is regular. Assume also without loss of generality that $0 \notin D$. Dente by D_1 the open image of D under the map $z \to 1/z$. Then it is known that for $x, y \in D$

$$G_D(x, y) = G_{D_1}\left(\frac{1}{x}, \frac{1}{y}\right) \tag{20}$$

where G_{D_1} is the Green function for D_1. ∞ is regular for D iff 0 is regular for D_1. It is not difficult to show that for any $x \in D$, the image under the map $z \to 1/z$ of the harmonic measure $P_D(x, dz)$ relative to D is the harmonic measure $P_{D_1}(1/x, dz)$ relative to D_1. From (20):

$$G_D(x,y) = K\left(\frac{1}{x} - \frac{1}{y}\right) - E_{\frac{1}{x}}\left| K\left(x_{T_1} - \frac{1}{y}\right)\right| + a_1\left(\frac{1}{x}\right)$$

where a_1 corresponds to D_1 as in Lemma 8 and T_1 is the exit time after zero from D_1:

$$= K\left(\frac{1}{x} - \frac{1}{y}\right) + \int \log\left|\frac{1}{z} - \frac{1}{y}\right| P_D(x, dz) + a_1\left(\frac{1}{x}\right)$$

$$= K(x-y) + \log|xy| - E_x[K(x_T - y)] - \int \log|yz| P_D(x, dz) + a_1\left(\frac{1}{x}\right)$$

$$= K(x-y) - E_x[K(x_T - y)] + \log|x| + \int \log|z| P_{D_1}\left(\frac{1}{x}, dz\right) + a_1\left(\frac{1}{x}\right).$$

Since 0 is regular for D_1, putting $y_0 = 0$ in (18) we get

$$-E_{\frac{1}{x}}[\log|x_{T_1}|] = -\log\frac{1}{|x|} + a_1\left(\frac{1}{x}\right).$$

This shows that

$$G_D(x, y) = K(x-y) - E_x[K(x_T - y)],$$

and that proves (19). Since $a \equiv 0$, (19)′ follows from (18). That completes the proof.

Concluding remarks

For the validity of Th. 10, the regularity of ∞ for the domain must be checked. Since the Green function cannot in general be given explicitly, the definition of regularity given above is not very useful. Of course, we could invert in a circle and check regularity this way. Several sufficient conditions for regularity are known. Simplest geometrical condition for regularity is the following well-known criterion:

PROPOSITION 11. *If $a \in \partial D$ is contained in a continuum contained in the complement of D, then a is regular for D.*

We present a short proof here. For other proofs see Heinz ([1] p. 93) and Helms ([2] p. 216).

PROOF OF PROPOSITION 11. Let $a \in F \subset \mathbf{R}^2 \setminus D$ be a continuum. If r is small, $F \cap (x : |x - a| \geqq r) \neq \emptyset$. Let D_1 be the open set

$$D_1 = (x : |x - a| < r) \setminus F.$$

D_1 is bounded open and ∂D_1 is connected. D_1 is then simply connected. By the Riemann mapping theorem, there is a 1-1 holomorphic map f defined on D_1 onto the open unit disc. Put $h(x) = 1 - |f(x)|$. h has the following properties: h is positive, h is superharmonic on D and $\lim_{x \to \partial D_1} h(x) = 0$, i.e., h is a "barrier". Every point $x_0 \in \partial D_1$ is then regular for D_1. If not, let $x_0 \in \partial D$ and assume that $P_{x_0}(T > 0) = 1$ where $T =$ exit time after zero from D_1.

$$E_{x_0}[h(x_{t+s}) : t + s < T] = E_{x_0}[E_{x_t}(h(x_s) : s < T) : t < T]$$

$$\leqq E_{x_0}[h(x_t) : t < T]$$

because relative to P_b for each $b \in D_1$, h is excessive. Letting $t \to 0$ and recalling that $\lim h(y) = 0$ as $D_1 \ni y \to x_0$, we get $\lim_{t \to 0} E_{x_0}[h(x_{t+s}) : t + s < T] = 0$. On the other hand,

$$\lim_{t \to 0} E_{x_0}[h(x_{t+s}) : t + s < T] \geqq E_{x_0}\left[\liminf_{t \to 0} h(x_{t+s})_{(t+s<T)}\right]$$

$$= E_{x_0}[h(x_s) : s < T] > 0$$

since for $s < T$, $x_s \in D$ and $h > 0$ on D_1. This contradiction shows that every point in ∂D_1 is regular. In particular, a is regular relative to D_1 and thus clearly implies regularity of a relative to D. That completes the proof.

It has thus been shown that for a half space, a strip, a wedge or in fact any simply connected domain D whose complement with respect to the Riemann sphere is more than one point the formula

$$G_D(x, y) = E_x[\log|x_T - y|] - \log|x - y|$$

obtains. The above discussion shows the following: If h is harmonic in D, continuous on $\bar{D}$, and $h(x) = O(\log |x|)$ as $|x| \to \infty$, then $h(x) = E_x[h(x_T)]$, provided of course that ∞ is regular for D.

The second assertion of Theorem 10 relates to Dirichlet problem. We shall consider this a little more generally.

THEOREM 12. *If m is a finite measure which does not charge D or the set of irregular points in ∂D and s the superharmonic function given by*

$$s(x) = \int K(x-y)\, m(dy) \tag{21}$$

then $E_x[|s(x_T)|] < \infty$ for all $x \in D$ and

$$s(x) = E_x[s(x_T)] - a(x).$$

PROOF. Let us assume that m is a probability measure such that the function in (21) is superharmonic on the plane. We shall show that

$$\int E_x[|K(x_T - y)|]\, m(dy) < \infty, \quad x \in D. \tag{22}$$

Assume (22) for the moment and suppose that m charges neither D nor the set of irregular points in ∂D. s is then harmonic in D. By (18)

$$\begin{aligned} s(x) &= \int K(x-y)\, m(dy) \\ &= \int (E_x[K(x_T - y)] - a(x))\, m(dy) \\ &= E_x[s(x_T)] - a(x) \end{aligned}$$

because by (22) use of Fubini theorem is permitted.

The proof of (22) runs as follows. For some $x_0 \in D$, $\int |K(x_0 - y)|\, m(dy) < \infty$. Therefore, there is no loss in generality in assuming that $\int |K(y)|\, m(dy) < \infty$ and that the closed unit disc is contained in D. If D_1 is the image of $D \setminus \{0\}$ under the map $z \to 1/z$, ∂D_1 is compact. Note that m cannot charge the origin. Let n be the image of m under the map $z \to 1/z$. Just as in the proof of Theorem 10 for $0 \neq x \in D$

$$\begin{aligned} &\int E_x[|K(x_T - y)|]\, m(dy) \\ &= \int E_x\left[\left|K\left(x_T - \frac{1}{y}\right)\right|\right] n(dy) \\ &= \int P_{D_1}\left(\frac{1}{x}, dz\right) \int \left|K\left(\frac{1}{z} - \frac{1}{y}\right)\right| n(dy) \end{aligned}$$

$$\leqq \int n(dy) \int |K(z-y)| P_{D_1}\left(\frac{1}{x}, dz\right) + \int \left|\log|z|\right| P_{D_1}\left(\frac{1}{x}, dz\right) + \int \left|\log|y|\right| n(dy),$$

$1/x \in D_1$ and by Lemma 7, $\int |\log|z|| P_{D_1}(1/x, dz) < \infty$. By assumption $\infty > \int |\log|y|| m(dy) = \int |\log|y|| n(dy)$. It remains to show that

$$\int E_x[|K(x_T - y)|] n(dy) < \infty, \quad x \in D_1,$$

where T_1 is the exit time from D_1. ∂D_1 being compact $|K(z-y)|$ behaves like $|K(y)|$ for large $|y|$ uniformly for $z \in \partial D$. We may thus assume that n has compact support. But then $K(z-y)$ is bounded below as z varies in ∂D_1 and y in the support of n. Fubini is then applicable and we need only show that

$$\int E_x[K(x_{T_1} - y)] n(dy) = E_x\left[\int K(x_{T_1} - y) n(dy)\right]$$

is finite. By Lemma 7 it is sufficient to show: u is super-harmonic in a neighbourhood of a bounded open set D, then $E_x[u(x_T)] < \infty$ for all $x \in D$, $T =$ exit time from D. $E_x \cdot [u(x_T)]$ is in fact harmonic in D. That completes the proof of the theorem.

For the sake of completeness let us look at the case when ∂D is compact and non-polar. In this case, the function a is given by

$$a(x) = \int \log|x-y| \mu(dy) + \log \frac{1}{c} \tag{23}$$

where c is the (logarithmic) capacity and μ the equilibrium distribution for the complement of D; in other words, μ solves the Robin's problem for the complement of D. To see this let y tend to infinity in (15). (Since x_T belongs to ∂D which is compact) we get

$$G(x, \infty) = a(x) \qquad x \in D. \tag{24}$$

There is a sequence x_n tending to ∞ for which the measures $P_{x_n}(x_T \in dz)$ converge weakly to a probability measure μ on ∂D. In particular for each $y \in D$, $\lim_{n\to\infty} E_{x_n}[\log|x_T - y|] = \int \log|z-y| \mu(dz)$.

Now let x tend to infinity in (15) along the sequence x_n. We obtain from (24)

$$a(y) = \int \log|z-y| \mu(dz) + \lim_n [K(x_n - y) + a(x_n)].$$

Clearly the last term in the above equation is independent of y. (23) has thus been proved. It is interesting to compare the above with the results in §14 p. 140–148 of [4].

ACKNOWLEDGEMENT

We take this opportunity to thank S. E. Graversen for encouragement and constructive criticism. We thank the referee for the references [3] and [4]. Their connections with the present work have been pointed out.

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